Chapter 5

Stability

Finite Difference Method

First Session Contents:

1) **Dependence of solution on data**
2) **Separation Variables Method (Fourier Method) - Continuous**
3) **Approximate Solution by Finite Difference Method - Discrete**

Continuous Dependence on Data

The solution is said to depend continuously on the data function *f(x)* provided there exists a constant C independent of f such that

$$\|u(x,t)\| \le C\|f(x)\|, \quad 0 \le t \le T$$

$\| \quad \|$ denotes a general form that could be

L^∞ the maximum norm $\|f(x)\| = \max|f(x)|, \quad 0 \le x \le 1$

L^2 or the energy norm $\|f(x)\| = \left\{\int_0^1 f(x)^2 dx\right\}^{\frac{1}{2}}$

Continuous Dependence on Data

For better understanding of continuous dependence on data, consider the following perturbation problem

$$\begin{aligned} &u_t - \alpha u_{xx} = 0 \\ &u(x,0) = f(x), \\ &u(0,t) = u(1,t) = 0, \end{aligned} \qquad \begin{aligned} &u_t - \alpha u_{xx} = 0, \\ &u(x,0) = f(x) + e(x), \\ &u(0,t) = u(1,t) = 0, \end{aligned}$$

$u(x,t)$ Solution of main problem

$\hat{u}(x,t)$ Solution of perturbed problem

The solution depends continuously on the data if

$$\|u(x,t) - \hat{u}(x,t)\| \le C\|e(x)\|, \quad 0 \le t \le T$$

If we think of *e(x)* as an error in representing the initial condition , then continuous dependence on the data implies that small errors in the data result in small changes in the solution.

Fourier Method for Differential Equations

Example

$$\begin{aligned} &L[u] = u_t - \alpha u_{xx} = 0, && 0 < x < 1, \quad 0 \le t \le T \quad \alpha = 1 \\ &u(x,0) = f(x), && 0 < x < 1 \\ &u(0,t) = u(1,t) = 0, && 0 \le t \le T \end{aligned}$$

Solution (Continuous):

Step 1 Separation Variables Method $u(x,t) = v(x)w(t)$

then

$u_t = vw' \quad , \quad u_{xx} = v''w$

thus

$$\left| \begin{aligned} &vw' - v''w = 0 \\ &\frac{w'}{w} = \frac{v''}{v} = \text{ثابت} \end{aligned} \right.$$

Separation Variables Method

Step 2 Boundary Conditions

$$\left|\begin{array}{l} v(0) = v(1) = 0 \\ v \neq 0 \end{array}\right. \longrightarrow \boxed{v'' = -\lambda^2 v, \quad 0 < x < 1, \quad v(0) = v(1) = 0}$$

From which it follows that:

$$\boxed{\lambda_n^2 = (n\pi)^2, \quad v_n(x) = \sin n\pi x, \quad n = 1, 2, \ldots}$$

Separation Variables Method

Step 3 Determining the transient behavior

From step 1 and step 2 $\longrightarrow$ $\frac{w'}{w} = -(n\pi)^2, \quad n = 1, 2, \ldots$

Thus: $\boxed{w_n(t) = b_n e^{-(n\pi)^2 t}, \quad n = 1, 2, \ldots}$

Step 4 Linear combination

$$u(x,t) = v_n(x) w_n(t) = \sum_{n=1}^{\infty} b_n e^{-(n\pi)^2 t} \sin n\pi x$$

Separation Variables Method

Step 5 Initial Condition

$$\left|\begin{array}{l} u(x,t) = \sum_{n=1}^{\infty} b_n e^{-(n\pi)^2 t} \sin n\pi x \\ u(x,0) = f(x) \end{array}\right. \longrightarrow f(x) = \sum_{n=1}^{\infty} b_n \sin n\pi x, \quad 0 < x < 1$$

From Sturm Liouville $\longrightarrow$ $b_n = 2\int_0^1 f(x) \sin n\pi x dx$

Finally

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} b_n e^{-(n\pi)^2 t} \sin n\pi x, \quad b_n = 2\int_0^1 f(x) \sin n\pi x dx}$$

Approximate Solution by Finite Difference Method

Example

$$\begin{array}{lll} L[u] = u_t - \alpha u_{xx} = 0, & 0 < x < 1, & 0 \le t \le T \\ u(x,0) = f(x), & 0 < x < 1 & \\ u(0,t) = u(1,t) = 0, & 0 \le t \le T & \end{array}$$

$$\left|\begin{array}{ll} x_i = i/(N+1) \quad i = 0, 1, \ldots, N+1 & h = \frac{1}{N+1} \\ t_n = (n-1)k, \quad n = 1, 2, \ldots, M, & k = \frac{T}{M-1} \end{array}\right.$$

Solution (Discrete):

Forward Time Central Space (FTCS) $\longrightarrow$

$$\left|\begin{array}{ll} \frac{U_i^{n+1} - U_i^n}{\Delta t} = \frac{U_{i-1}^n - 2U_i^n + U_{i+1}^n}{(\Delta x)^2}, & i = 1, 2, \ldots, N \\ & n = 0, 1, \ldots \\ U_i^0 = f_i, \quad i = 1, 2, \ldots, N & \\ U_0^n = U_{N+1}^n = 0, \quad n = 1, 2, \ldots & \end{array}\right.$$

Approximate Solution by Finite Difference Method

Step 1 Separation index

if $U_i^n = V_i W^n$

From which it follows that:

$$V_i \frac{W^{n+1} - W^n}{\Delta t} = \frac{V_{i-1} - 2V_i + V_{i+1}}{(\Delta x)^2} W^n$$

or $$\boxed{\frac{(\Delta x)^2}{\Delta t}\frac{W^{n+1} - W^n}{W^n} = \frac{V_{i-1} - 2V_i + V_{i+1}}{V_i} = \text{ثابت} = -\lambda}$$

Comparison whit Separation Variables Method $\boxed{\frac{w'}{w} = \frac{v''}{v} = \text{ثابت}}$

Approximate Solution by Finite Difference Method

Step 2 Boundary Conditions

$$\begin{cases} V_0 = 0 = V_{N+1} \\ \dfrac{(\Delta x)^2}{\Delta t}\dfrac{W^{n+1} - W^n}{W^n} = \dfrac{V_{i-1} - 2V_i + V_{i+1}}{V_i} = \text{ثابت} = -\lambda \end{cases}$$

thus $$\boxed{-V_{i-1} + 2V_i - V_{i+1} = \lambda V_i,\ \ i = 1, 2, \ldots, N,\ V_0 = V_{N+1} = 0}$$

or

$$\boxed{A\vec{V} = \lambda\vec{V}} \longrightarrow A = \begin{bmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & \ddots & \ddots \\ & & & -1 & 2 & -1 \\ & & & & -1 & 2 \end{bmatrix} \qquad \vec{V} = [V_1, V_2, V_3, \ldots, V_{N-1}, V_N]^T$$

Approximate Solution by Finite Difference Method

Determining the eigenvalues

eigenvalues Problem $$\begin{cases} A = \begin{bmatrix} b & c & & & \\ a & b & c & & \\ & a & b & c & \\ & & \ddots & \ddots & \ddots \\ & & & a & b & c \\ & & & & a & b \end{bmatrix} \\ \text{if} \quad V_0 = 0 = V_{N+1} \end{cases}$$

$$A\vec{V} = \lambda\vec{V} \longrightarrow \boxed{aV_{i-1} + bV_i + cV_{i+1} = \lambda V_i, \quad i = 1, 2, \ldots, N}$$

Approximate Solution by Finite Difference Method

Continue

$$\begin{cases} ar^{i-1} + (b-\lambda)r^i + cr^{i+1} = 0 \\ \text{or} \\ cr^2 + (b-\lambda)r + a = 0 \end{cases} \longrightarrow V_i = \alpha r_1^i + \beta r_2^i$$

From BC $V_0 = V_{N+1} = 0$ $$\longrightarrow \begin{cases} \alpha + \beta = 0 \\ \alpha r_1^{N+1} + \beta r_2^{N+1} = 0 \end{cases} \longrightarrow \boxed{\frac{r_1}{r_2} = e^{j2k\pi/(N+1)}, \quad k = 1, 2, \ldots, N}$$

So $$\begin{cases} \dfrac{r_1}{r_2} = e^{j2k\pi/(N+1)}, \quad k = 1, 2, \ldots, N \\ r_1 r_2 = \dfrac{a}{c} \quad (\text{From } cr^2 + (b-\lambda)r + a = 0\ ) \end{cases} \longrightarrow \begin{cases} r_2 = \sqrt{\dfrac{a}{c}}e^{-jk\pi/(N+1)} \\ r_1 = \sqrt{\dfrac{a}{c}}e^{jk\pi/(N+1)} \end{cases}$$

Approximate Solution by Finite Difference Method

Continue

$$\left|\begin{array}{l} r_1 = \sqrt{\dfrac{a}{c}}e^{jk\pi/(N+1)} \\ r_2 = \sqrt{\dfrac{a}{c}}e^{-jk\pi/(N+1)} \\ r_1 + r_2 = -\dfrac{b-\lambda}{c} \quad \text{(From } cr^2 + (b-\lambda)r + a = 0 \text{)} \end{array}\right. \longrightarrow \boxed{\lambda_k = b + 2c\sqrt{\frac{a}{c}}\cos\frac{k\pi}{N+1}, \qquad k = 1,2,\dots,N}$$

$$\left|\begin{array}{l} V_i = \alpha r_1^i + \beta r_2^i \\ r_1 = \sqrt{\dfrac{a}{c}}e^{jk\pi/(N+1)} \\ r_2 = \sqrt{\dfrac{a}{c}}e^{-jk\pi/(N+1)} \end{array}\right. \longrightarrow \boxed{V_i = 2\alpha j\left(\sqrt{\frac{a}{c}}\right)^i \sin\frac{ik\pi}{N+1}}$$

Approximate Solution by Finite Difference Method

Thus

$$V_i^k = \left(\sqrt{\frac{a}{c}}\right)^i \sin\frac{ik\pi}{N+1} \longrightarrow \left|\begin{array}{l} \lambda_k = 2 - 2\cos\dfrac{k\pi}{N+1}, \qquad k = 1,2,\dots,N \qquad \text{(Eigenvectors)} \\ \vec{V}^k = \begin{bmatrix} \sin k\pi x_1 \\ \sin k\pi x_2 \\ \vdots \\ \sin k\pi x_N \end{bmatrix}, \quad k = 1,2,\dots,N \qquad \text{(Eigenvalues)} \end{array}\right.$$

also
$$\left|\begin{array}{l} \vec{V}^{k_1} \cdot \vec{V}^{k_2} = 0 \\ \vec{V}^{k} \cdot \vec{V}^{k} = \dfrac{N}{2} \end{array}\right.$$

Approximate Solution by Finite Difference Method

Step 3 Determining the transient behavior

$$\frac{(\Delta x)^2}{\Delta t}\frac{W^{n+1} - W^n}{W^n} = \frac{V_{i-1} - 2V_i + V_{i+1}}{V_i} = \text{ثابت} = -\lambda \longrightarrow \frac{(\Delta x)^2}{\Delta t}\frac{W_k^{n+1} - W_k^n}{W_k^n} = -\lambda_k$$

thus
$$\boxed{W_k^{n+1} = (1 - r\lambda_k)W_k^n, \quad r = \frac{\Delta t}{(\Delta x)^2}, \qquad k = 1,2,\dots,N}$$

If W_k^0 is known $\longrightarrow W_k^n = (1 - r\lambda_k)^{n-1}W_k^0, \quad n = 1,2,\dots$

Step 4 Linear combination

$$U_i^n = \sum_{k=1}^{N} W_k^n \sin k\pi x_i, \qquad \begin{array}{l} i = 1,2,\dots,N \\ n = 1,2,\dots \end{array} \qquad \left|\begin{array}{l} W_k^n = (1 - r\lambda_k)^{n-1}W_k^0 \\ \vec{V}^k = \sin k\pi x_i \end{array}\right.$$

$$\longrightarrow \boxed{\vec{U}^n = \sum_{k=1}^{N} W_k^0(1 - r\lambda_k)^n \vec{V}^k}$$

Approximate Solution by Finite Difference Method

Step 5 Initial Condition

Considering $f = [f_1, f_2, \dots, f_N]^T$ and $U_i^0 = f_i, \quad i = 1,2,\dots,N$

thus
$$f = \sum_{k=1}^{N} W_k^0 \vec{V}^k$$

Orthogonally principle $\longrightarrow \boxed{W_k^0 = \dfrac{\vec{f} \cdot \vec{V}^k}{\vec{V}^k \cdot \vec{V}^k}, \quad k = 1,2,\dots,N}$

Comparison whit Separation Variables Method $\boxed{b_n = 2\int_0^1 f(x)\sin n\pi x\,dx}$

Approximate Solution by Finite Difference Method

Step 5 Initial Condition

Step 3 (From Continuous) $w_n(t) = b_n e^{-(n\pi)^2 t}, \quad n = 1, 2, \dots$

$$\longrightarrow \frac{W_k(t+\Delta t)}{W_k(t)} = e^{-(k\pi)^2 \Delta t} = 1 - (k\pi)^2 \Delta t + \cdots \quad \text{(Maclaurin Expansion)}$$

Step 3 (From Discrete) $\dfrac{W_k^{n+1}}{W_k^n} = 1 - r\lambda_k = 1 - \dfrac{\Delta t}{(\Delta x)^2}\lambda_k$

also $\lambda_k = 2 - 2\cos\dfrac{k\pi}{N+1} = \left[\dfrac{k\pi}{N+1}\right]^2 + \cdots, \quad \Delta x = \dfrac{1}{N+1}$

Thus: $\boxed{\dfrac{W_k^{n+1}}{W_k^n} = 1 - [k\pi]^2 \Delta t + \cdots}$

17

Example

$$u_t - u_{xx} = 0, \quad 0 < x < 1, \quad t > 0$$
$$u(x,0) = x^2, \quad 0 < x < 1$$
$$u(0,t) = u(1,t) = 0, \quad t > 0$$

if
$$\left|\begin{array}{l} \{x_1, x_2, x_3, x_4, x_5\} = \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\} \\ \Delta x = 1/4 \\ N = 3 \\ f_i = x_i^2 \end{array}\right. \longrightarrow \boxed{f_1 = \frac{1}{16}, \quad f_2 = \frac{1}{4}, \quad f_3 = \frac{9}{16}}$$

Also
$$\vec{V}^1 = \begin{bmatrix} \sin\frac{\pi}{4} \\ \sin\frac{\pi}{2} \\ \sin\frac{3\pi}{4} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ 1 \\ \frac{\sqrt{2}}{2} \end{bmatrix}, \quad \vec{V}^2 = \begin{bmatrix} \sin\frac{2\pi}{4} \\ \sin\frac{2\pi}{2} \\ \sin\frac{6\pi}{4} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$
$$\vec{V}^3 = \begin{bmatrix} \sin\frac{3\pi}{4} \\ \sin\frac{3\pi}{2} \\ \sin\frac{9\pi}{4} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ -1 \\ \frac{\sqrt{2}}{2} \end{bmatrix}$$

18

Example

Also
$$\left|\begin{array}{l} W_1^0 = \dfrac{\vec{f}\cdot\vec{V}^1}{\vec{V}^1\cdot\vec{V}^1} = \dfrac{\frac{\sqrt{2}}{32} + \frac{1}{4} + \frac{9\sqrt{2}}{32}}{\frac{1}{2} + 1 + \frac{1}{2}} = \dfrac{4 + 5\sqrt{2}}{32} \\ W_2^0 = \dfrac{\vec{f}\cdot\vec{V}^2}{\vec{V}^2\cdot\vec{V}^2} = \dfrac{\frac{1}{16} + 0 - \frac{9}{16}}{1 + 0 + 1} = -\dfrac{1}{4} \\ W_3^0 = \dfrac{\vec{f}\cdot\vec{V}^3}{\vec{V}^3\cdot\vec{V}^3} = \dfrac{\frac{\sqrt{2}}{32} - \frac{1}{4} + \frac{9\sqrt{2}}{32}}{\frac{1}{2} + 1 + \frac{1}{2}} = \dfrac{-4 + 5\sqrt{2}}{32} \end{array}\right.$$

From $\lambda_k = 2 - 2\cos\dfrac{k\pi}{N+1} = \left[\dfrac{k\pi}{N+1}\right]^2 + \cdots, \xrightarrow{N=3} \boxed{\lambda_1 = 2 - \sqrt{2}, \quad \lambda_2 = 2, \quad \lambda_3 = 2 + \sqrt{2}}$

Thus $\boxed{U^n = W_1^0(1 - r\lambda_1)^n \vec{V}^1 + W_2^0(1 - r\lambda_2)^n \vec{V}^2 + W_3^0(1 - r\lambda_3)^n \vec{V}^3}$

19